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Formal and analytic first integrals of the Einstein–Yang–Mills equations

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Abstract
In this paper we provide a complete description of the first integrals of the classical Einstein–Yang–Mills equations that can be described by formal series. As a corollary we also obtain a complete description of the analytic first integrals in a neighbourhood of the origin.

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1. Introduction to the problem

The static, spherically symmetric Einstein–Yang–Mills equations [1–3, 12, 13] with a cosmological constant $a \in \mathbb{R}$ are given by the differential system

$$\begin{aligned} \dot{r} &= rN, & \dot{W} &= rU, \\ \dot{N} &= (k-N)N - 2U^2, & \dot{k} &= s(1-2ar^2) + 2U^2 - k^2, \\ \dot{U} &= sWT + (N-k)U, & \dot{T} &= 2UW - NT, \end{aligned} \tag{1}$$

where $r, W, N, k, U, T \in \mathbb{R}^6$ and $s \in \{-1; 1\}$, and the dot denotes a derivative with respect to the space–time variable t.

Let

$$f = 2kN - N^2 - 2U^2 - s(1 - T^2 - ar^2). \tag{2}$$

Then, over the solutions $(r(t), W(t), N(t), k(t), U(t), T(t))$ of system (1) it holds

$$\frac{\mathrm{d}f(t)}{\mathrm{d}t} = -2N(t)f(t).$$

Therefore, we obtain that $f = 0$ is an invariant hypersurface under the flow of system (1); i.e., if a solution of system (1) has a point on $f = 0$ the whole solution is contained in $f = 0$.

For physical reasons (see again [2]), it is interesting to study the solutions of (1) over the hypersurface $f = 0$. Hence, from (2), the solutions of system (1) on $f = 0$ satisfy the equality

$$s(1 - ar^2) = 2kN - N^2 + sT^2 - 2U^2. \tag{3}$$

Thus, defining the variables $x_1 = r, x_2 = W, x_3 = N, x_4 = k, x_5 = U, x_6 = T$, and taking into account (3), we obtain that system (1) on $f = 0$ is equivalent to the homogeneous polynomial differential system

$$\begin{aligned}
\dot{x}_1 &= X_1(x_1, \ldots, x_6) = x_1x_3,\\
\dot{x}_2 &= X_2(x_1, \ldots, x_6) = x_1x_5,\\
\dot{x}_3 &= X_3(x_1, \ldots, x_6) = (x_4 - x_3)x_3 - 2x_5^2,\\
\dot{x}_4 &= X_4(x_1, \ldots, x_6) = -(x_4 - x_3)^2 + s\left(-ax_1^2 + x_6^2\right),\\
\dot{x}_5 &= X_5(x_1, \ldots, x_6) = sx_2x_6 + (x_3 - x_4)x_5,\\
\dot{x}_6 &= X_6(x_1, \ldots, x_6) = 2x_2x_5 - x_3x_6,
\end{aligned} \tag{4}$$

of degree 2 in $\mathbb{R}^6$ depending on the real parameter a. Furthermore, taking into account that s is a constant, we can rewrite the hypersurface $f = 0$ in the new variables as $F = s$, where

$$F = 2x_3x_4 - x_3^2 + s\left(ax_1^2 + x_6^2\right) - 2x_5^2. \tag{5}$$

Clearly, by construction, F is a homogeneous first integral of degree 2 of system (4). Furthermore, it is easy to obtain that

$$G = x_2^2 - x_1x_6 \tag{6}$$

is another homogeneous first integral of degree 2 of system (4).

The aim of this paper is to study the existence of formal series first integrals of system (4) that are different from F and G. The use of formal series in the study of differential equations and, in particular, in the existence of their first integrals is a classical tool. Indeed, for instance, solutions described by formal series around singularities have been studied by Seidenberg [11], the existence of first integrals given by formal series has been studied by Nemytskii and Stepanov [10], Moussu [9], ... However, the greatest success in using formal series to study differential equations has been achieved by Écalle [5] who used them to prove Dulac's conjecture.

A *formal series first integral* $f = f(x_1, \ldots, x_6)$ of system (4) is a formal power series in the variables $x_1, \ldots, x_6$ such that

$$\sum_{k=1}^{6} \frac{\partial f}{\partial x_k} X_k(x_1, \ldots, x_6) = 0. \tag{7}$$

In what follows when we talk about formal power series we only say formal series.

The first main result of this paper is

Theorem 1. *All formal series first integrals of system (4) are formal series in the variables F and G.*

Here an *analytic first integral* of system (4) is an analytic function which is constant over the trajectories of system (4). The second main result of this paper is

Theorem 2. *All analytic first integrals of the system (4) in a neighbourhood of the origin are analytic functions in the variables F and G.*

These kinds of integrability studied in this paper for the EYM system have been considered by many authors in very similar interesting physical systems, such as for instance the Bianchi IX system or the mixmaster model: see, for instance, [4, 6, 8].

The paper is organized as follows. In section 2 we state some preliminary results that will be used throughout the paper. In section 3 we study the formal series first integrals of the EYM system (4) restricted to $x_1 = x_2 = 0$, and in section 4 we provide the proof of theorems 1 and 2.

The method for proving these theorems is general and can be used in many other differential systems (mainly polynomial differential systems) for studying their integrability.

2. Preliminary results

We state some results that we shall use later on.

Lemma 3. *Let x and y be one-dimensional variables. Given a formal series $f(x)$, there exists a formal series $g(x, y)$ such that*

$$f(x) + f(y) = f(x + y) + f(0) - xyg(x, y).$$

Proof. We write $f(z) = f(0) + \sum_{j=1}^{\infty} f_j z^j$. Then, using Newton's binomial formula,

$$\begin{aligned} f(x+y) + f(0) &= 2f(0) + \sum_{j=1}^{\infty} f_j (x+y)^j = 2f(0) + \sum_{j=1}^{\infty} f_j \sum_{k=0}^{j} \binom{j}{k} x^k y^{j-k} \\ &= f(x) + f(y) + xy \sum_{j=1}^{\infty} f_j \sum_{k=0}^{j-2} \binom{j}{k+1} x^k y^{j-2-k} \\ &= f(x) + f(y) + xyg(x, y). \end{aligned}$$

□

Lemma 4. *Let x_k be one-dimensional variables for $k = 1, \ldots, n$ with $n > 1$. Let $f = f(x_1, \ldots, x_n)$ be a formal series such that in $x_l = x_j$, $j \neq l$, $j, l \in \{1, \ldots, n\}$, $f(x_1, \ldots, x_n)|_{x_l = x_j} = \overline{f}$, where $\overline{f}$ is a formal series in the variables $x_1, \ldots, x_{l-1}, x_{l+1}, \ldots, x_n$. Then, there exists a formal series $g = g(x_1, \ldots, x_n)$ such that $f = \overline{f} + (x_l - x_j)g$.*

Proof. We denote by $\mathbb{Z}^+$ the set of all non-negative integers. We write

$$f = \sum_{(k_1,\ldots,k_n)\in(\mathbb{Z}^+)^n} f_{k_1,\ldots,k_n} x_1^{k_1} \cdots x_n^{k_n}.$$

Without loss of generality we can assume $l = 1$ and $j = 2$. Then, writing $x_1 = x_2 + (x_1 - x_2)$, and using Newton's binomial formula, we have

$$\begin{aligned} f &= \sum_{(k_1,\ldots,k_n)\in(\mathbb{Z}^+)^n} f_{k_1,\ldots,k_n} (x_2 + (x_1 - x_2))^{k_1} x_2^{k_2} \cdots x_n^{k_n} \\ &= \sum_{(k_1,\ldots,k_n)\in(\mathbb{Z}^+)^n} f_{k_1,\ldots,k_n} \sum_{j=0}^{k_1} \binom{k_1}{j} x_2^j (x_1 - x_2)^{k_1 - j} x_2^{k_2} \cdots x_n^{k_n} \\ &= \sum_{(k_1,\ldots,k_n)\in(\mathbb{Z}^+)^n} f_{k_1,\ldots,k_n} x_2^{k_1} x_2^{k_2} \cdots x_n^{k_n} \end{aligned}$$

$$+ (x_1 - x_2) \sum_{(k_1,\ldots,k_n)\in(\mathbb{Z}^+)^n} f_{k_1,\ldots,k_n} \sum_{j=0}^{k_1-1} \binom{k_1}{j} x_2^j (x_1 - x_2)^{k_1-j-1} x_2^{k_2} \cdots x_n^{k_n}$$

$$= f(x_2, x_2, \ldots, x_n) + (x_1 - x_2) g(x_1, \ldots, x_n) = \overline{f} + (x_1 - x_2) g,$$

which finishes the proof of the lemma. □

Let τ and σ be defined by

$$\tau : (x_1, x_2, x_3, x_4, x_5, x_6, t) \to (-x_1, x_2, x_3, x_4, -x_5, -x_6, t),$$
$$\sigma : (x_1, x_2, x_3, x_4, x_5, x_6, t) \to (x_1, -x_2, -x_3, -x_4, x_5, x_6, -t),$$

and note that system (4) is invariant by these two symmetries.

Proposition 5. *Let $g = g(x_1, x_2, x_3, x_4, x_5, x_6)$ be a formal series first integral of system (4). Then,*

(i) $f = (g \cdot \tau(g)) \cdot \sigma(g \cdot \tau(g))$ is another first integral of system (4) invariant by τ and σ;
(ii) the monomials of f are of the form $x_1^{l_1} x_2^{l_2} x_3^{l_3} x_4^{l_4} x_5^{l_5} x_6^{l_6}$ with $l_1 + l_5 + l_6$ and $l_2 + l_3 + l_4$ even.

Proof. The first statement of the proposition follows taking into account that system (4) is invariant under τ and σ, and that $\tau^2 = \sigma^2 = \mathrm{Id}$.

To prove the second statement we write f in formal series

$$f = \sum_{(l_1,l_2,l_3,l_4,l_5,l_6)\in(\mathbb{Z}^+)^6} f_{l_1,l_2,l_3,l_4,l_5,l_6} x_1^{l_1} x_2^{l_2} x_3^{l_3} x_4^{l_4} x_5^{l_5} x_6^{l_6}.$$

Then, since $\tau(f) = f$, i.e., $f - \tau(f) = 0$, it holds

$$\sum_{(l_1,l_2,l_3,l_4,l_5,l_6)\in(\mathbb{Z}^+)^6} (1 - (-1)^{l_1+l_5+l_6}) f_{l_1,l_2,l_3,l_4,l_5,l_6} x_1^{l_1} x_2^{l_2} x_3^{l_3} x_4^{l_4} x_5^{l_5} x_6^{l_6} = 0,$$

which clearly implies that $l_1 + l_5 + l_6$ is even. In a similar way, using that $f - \sigma(f) = 0$ we get that $l_2 + l_3 + l_4$ is even. □

We shall need the following preliminary result. We consider the autonomous differential system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \qquad \mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{C}^n, \tag{8}$$

where $\mathbf{f}$ is a vector-valued function of dimension n satisfying $\mathbf{f}(\mathbf{0}) = \mathbf{0}$. As usual, $\mathbb{C}$ denotes the complex field. We denote by $\mathbf{A}$ the Jacobian matrix $\partial\mathbf{f}/\partial\mathbf{x}(\mathbf{0})$ of the vector field $\mathbf{f}(\mathbf{x})$ at $\mathbf{x} = \mathbf{0}$. Then, the following two results are proved in theorems 1 and 2 of [7], respectively. Although in [7], theorem 2 is stated for analytic series, its proof is also valid for formal series as stated in theorem 7.

Theorem 6. *Assume that the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of $\mathbf{A}$ satisfy the conditions: $\lambda_1 = 0$ and $\sum_{i=2}^n k_i\lambda_i \neq 0$ for any $k_i \in \mathbb{Z}^+$ and $\sum_{i=2}^n k_i \geqslant 1$. For $n > 2$, system (8) has a formal series first integral in a neighbourhood of $\mathbf{x} = \mathbf{0}$ if and only if the singular point $\mathbf{x} = \mathbf{0}$ is not isolated.*

Theorem 7. *If the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of $\mathbf{A}$ satisfy the conditions: $\sum_{i=1}^n k_i\lambda_i \neq 0$, for any $k_i \in \mathbb{Z}^+$ and $\sum_{i=1}^n k_i \geqslant 1$; then, system (8) does not have any formal series first integral in a neighbourhood of $\mathbf{x} = \mathbf{0}$.*

3. First integrals of system (4) restricted to $x_1 = x_2 = 0$

We consider the system (4) restricted to $x_1 = x_2 = 0$, that is

$$\begin{aligned} \dot{x}_3 &= (x_4 - x_3)x_3 - 2x_5^2, \qquad & \dot{x}_4 &= -(x_4 - x_3)^2 + sx_6^2, \\ \dot{x}_5 &= (x_3 - x_4)x_5, & \dot{x}_6 &= -x_3x_6. \end{aligned} \tag{9}$$

The objective of this section is to study the formal series first integrals of system (9).

Proposition 8. *The unique formal series first integrals of system (9) invariant by τ and σ are formal series in the variable $\overline{F}$, where $\overline{F} = x_3(2x_4 - x_3) - 2x_5^2 + sx_6^2$.*

To prove proposition 8 we introduce and prove some auxiliary results.

Lemma 9. *The unique formal series first integrals invariant by τ and σ of system (9) restricted to $x_5 = 0$ are formal series in the variable $\widetilde{F}$, where $\widetilde{F} = x_3(2x_4 - x_3) + sx_6^2$.*

Proof. Since $\widetilde{F}$ is a first integral of system (9), restricted to $x_5 = 0$, computing x_6 from $\widetilde{F}(x_3, x_4, x_6) = \widetilde{f}$, system (9) restricted to $x_5 = 0$ and to the level set $\widetilde{F} = \widetilde{f}$ becomes

$$\dot{x}_3 = (x_4 - x_3)x_3, \qquad \dot{x}_4 = \widetilde{f} - x_4^2. \tag{10}$$

We claim that system (10) has no formal series first integrals in a neighbourhood of the singular point $p = (x_3, x_4) = \sqrt{\widetilde{f}}(1, 1)$. Indeed, since the eigenvalues of the Jacobian matrix of system (10) at the singular point p are $-\sqrt{\widetilde{f}}$ and $-2\sqrt{\widetilde{f}}$, the claim follows immediately from theorem 7.

Now, we proceed by contradiction. Assume that system (9) has a formal series first integral H invariant by τ and σ which is not a formal series in $\widetilde{F}$. Repeating the arguments of the proof of proposition 5 for system (4) restricted to $x_1 = x_2 = x_5 = 0$, the restriction $\widetilde{H}$ of H to this system only contains monomials $x_3^{l_3}x_4^{l_4}x_6^{l_6}$ with l_6 even. Substituting into $\widetilde{H}$ the variable x_6 from $\widetilde{F}(x_3, x_4, x_6) = \widetilde{f}$ (that is, $x_6^2 = \widetilde{f} - x_3(2x_4 - x_3)$), then $\widetilde{H}$ becomes a formal series first integral of system (10), and consequently there is a formal series first integral defined in a neighbourhood of the singular point $p = (x_3, x_4) = \sqrt{\widetilde{f}}(1, 1)$, in contradiction with the statement of the claim proved above. □

Lemma 10. *The unique formal series first integrals invariant by τ and σ of system (9) restricted to $x_6 = 0$ are formal series in the variable $\widehat{F}$, where $\widehat{F} = x_3(2x_4 - x_3) - 2x_5^2$.*

Proof. Since $\widehat{F}$ is a first integral of system (9) restricted to $x_6 = 0$, computing x_5 from $\widehat{F}(x_3, x_4, x_5) = \widehat{f}$, system (9) restricted to $x_6 = 0$ and to the level set $\widehat{F} = \widehat{f}$ becomes

$$\dot{x}_3 = \widehat{f} - x_4x_3, \qquad \dot{x}_4 = -(x_4 - x_3)^2. \tag{11}$$

Then, since the eigenvalues of the Jacobian matrix of system (11) at the singular point $p = (x_3, x_4) = \sqrt{\widehat{f}}(1, 1)$ are 0 and $-\sqrt{\widehat{f}}$, from theorem 6 we get that system (11) does not have formal series first integrals in a neighbourhood of p. Now, proceeding as in the last part of the proof of lemma 9 we obtain the desired statement. □

Lemma 11. *Let $f = f(x_3, x_4)$ be a formal series satisfying*

$$(x_4 - x_3)\left(x_3\frac{\partial f}{\partial x_3} - (x_4 - x_3)\frac{\partial f}{\partial x_4}\right) = (ax_4 + bx_3)f \tag{12}$$

with $a, b \in \mathbb{Z}$, and $a + b \neq -n$ for all $n \in \mathbb{Z}^+$. Then, $f = 0$.

Proof. We proceed by contradiction. We assume $f \neq 0$ and we consider two different cases.

Case 1. f is not divisible by $x_4 - x_3$. In this case, we reach a contradiction from (12) taking into account that, by hypothesis, $x_4 - x_3$ does not divide $x_4 + bx_3$; otherwise $a + b = 0$.

Case 2. f is divisible by $x_4 - x_3$. In this case, we write $f = (x_4 - x_3)^m h$ with $m \geqslant 1$, and $h = h(x_3, x_4)$ is a formal series which is not divisible by $x_4 - x_3$. Furthermore, from (12), h satisfies

$$(x_4 - x_3)\left(x_3 \frac{\partial h}{\partial x_3} - (x_4 - x_3)\frac{\partial h}{\partial x_4}\right) = ((a + m)x_4 + bx_3).$$

Then, taking into account that by hypothesis $x_4 - x_3$ does not divide $(a+m)x_4 + bx_3$ (otherwise $a + b = -m$), we get that h must be divisible by $x_4 - x_3$, a contradiction. □

Proposition 12. *Let* $f = f(x_3, x_4, x_5, x_6)$ *be a formal series satisfying*

$$\left((x_4 - x_3)x_3 - 2x_5^2\right)\frac{\partial f}{\partial x_3} + \left(-(x_4 - x_3)^2 + sx_6^2\right)\frac{\partial f}{\partial x_4} + (x_3 - x_4)x_5\frac{\partial f}{\partial x_5} - x_3x_6\frac{\partial f}{\partial x_6} = lx_4 f, \tag{13}$$

where l is a positive integer. Then, $f = x_5x_6g$ *for some formal series* $g = g(x_3, x_4, x_5, x_6)$.

Proof. We write f as a formal series in the variable x_5, then

$$f = \sum_{k\geqslant 0} f_k x_5^k, \qquad \text{where} \quad f_k = f_k(x_3, x_4, x_6) \text{ are formal series.}$$

We first prove that $f_0 = 0$. Suppose that $f_0 \neq 0$ and we write f_0 as a formal series in the variables x_6, then

$$f_0 = \sum_{k\geqslant 0} f_{0,k} x_6^k, \qquad \text{where} \quad f_{0,k} = f_{0,k}(x_3, x_4) \text{ are formal series.}$$

We consider two different cases.

Case 1. f_0 is not divisible by x_6. In this case $f_{0,0} \neq 0$ and $f_{0,0}$ satisfies (13) restricted to $x_5 = x_6 = 0$; i.e., it satisfies equation (12) with $a = l \geqslant 1$ and $b = 0$. Then, from lemma 11 we get that $f_{0,0} = 0$, a contradiction.

Case 2. f_0 is divisible by x_6. In this case we write $f_0 = x_6^m g_0$ with $m \geqslant 1$ and $g_0 = g_0(x_3, x_4, x_6)$ is a formal series which is not divisible by x_6. Furthermore, from (13) restricted to $x_5 = 0$ we get that g_0 satisfies

$$(x_4 - x_3)x_3\frac{\partial g_0}{\partial x_3} + \left(-(x_4 - x_3)^2 + sx_6^2\right)\frac{\partial g_0}{\partial x_4} - x_3x_6\frac{\partial g_0}{\partial x_6} = (lx_4 + mx_3)g_0. \tag{14}$$

We write g_0 as a formal series in the variable x_6, then

$$g_0 = \sum_{k\geqslant 0} g_{0,k} x_6^k, \qquad \text{where} \quad g_{0,k} = g_{0,k}(x_3, x_4) \text{ are formal series.}$$

Then, since g_0 is not divisible by x_6, we have $g_{0,0} \neq 0$. However, $g_{0,0}$ satisfies (14) restricted to $x_6 = 0$. That is, it satisfies (12) with $a = l \geqslant 1$ and $b = m \geqslant 1$. Then, from lemma 11 we get $g_{0,0} = 0$, a contradiction.

Thus, $f_0 = 0$ and hence $f = x_5\overline{f}$ for some formal series $\overline{f} = \overline{f}(x_3, x_4, x_5, x_6)$ that satisfies, after dividing by x_5,

$$\begin{aligned}\left((x_4 - x_3)x_3 - 2x_5^2\right)&\frac{\partial \overline{f}}{\partial x_3} + \left(-(x_4 - x_3)^2 + sx_6^2\right)\frac{\partial \overline{f}}{\partial x_4}\\ &+ (x_3 - x_4)x_5\frac{\partial \overline{f}}{\partial x_5} - x_3x_6\frac{\partial \overline{f}}{\partial x_6} = ((l + 1)x_4 - x_3)\overline{f}.\end{aligned} \tag{15}$$

Now, we write $\overline{f}$ as a formal series in the variable x_6 then

$$\overline{f} = \sum_{k\geqslant 0} \overline{f}_k x_6^k, \qquad \text{where} \quad \overline{f}_k = \overline{f}_k(x_3, x_4, x_5) \text{ are formal series.}$$

The proof of the proposition will be finished if we prove that $\overline{f}_0 = 0$. Again we will proceed by contradiction. We assume $\overline{f}_0 \neq 0$ and we write it as a formal series in the variables x_5, then

$$\overline{f}_0 = \sum_{k\geqslant 0} \overline{f}_{0,k} x_5^k, \qquad \text{where} \quad \overline{f}_{0,k} = \overline{f}_{0,k}(x_3, x_4) \text{ are formal series.}$$

We consider two different cases.

Case a. $\overline{f}_0$ is not divisible by x_5. In this case $\overline{f}_{0,0} \neq 0$ and $\overline{f}_{0,0}$ satisfies (15) restricted to $x_5 = x_6 = 0$; i.e., it satisfies equation (12) with $a = l+1 \geqslant 2$ and $b = -1$. Then, from lemma 11 we get that $\overline{f}_{0,0} = 0$, a contradiction.

Case b. $\overline{f}_0$ is divisible by x_5. In this case we write $\overline{f}_0 = x_5^m \overline{g}_0$ with $m \geqslant 1$ and $\overline{g}_0 = \overline{g}_0(x_3, x_4, x_5)$ is a formal series which is not divisible by x_5. Furthermore, from (15) restricted to $x_6 = 0$ we get that $\overline{g}_0$ satisfies

$$\begin{aligned}\big((x_4 - x_3)x_3 - 2x_5^2\big)\frac{\partial \overline{g}_0}{\partial x_3} + \big(-(x_4 - x_3)^2 + s x_6^2\big)\frac{\partial \overline{g}_0}{\partial x_4} + (x_3 - x_5)x_5\frac{\partial \overline{g}_0}{\partial x_5} \\ = ((l + m + 1)x_4 - (m + 1)x_3)\overline{g}_0. \end{aligned} \tag{16}$$

We write $\overline{g}_0$ as a formal series in the variable x_5, then

$$\overline{g}_0 = \sum_{k\geqslant 0} \overline{g}_{0,k} x_5^k, \qquad \text{where} \quad \overline{g}_{0,k} = \overline{g}_{0,k}(x_3, x_4) \text{ are formal series.}$$

Then, since $\overline{g}_0$ is not divisible by x_5, we have $\overline{g}_{0,0} \neq 0$. However, $\overline{g}_{0,0}$ satisfies (16) restricted to $x_5 = 0$. That is, it satisfies (12) with $a = l + m + 1$ and $b = -m - 1$. Then, from lemma 11, we get $\overline{g}_{0,0} = 0$, a contradiction. □

Proof of proposition 8. Let f be a formal series first integral of system (9). Now, if we write f as a formal series in the variable x_5, then

$$f = \sum_{k\geqslant 0} \widetilde{f}_k x_5^k, \qquad \text{where} \quad \widetilde{f}_k = \widetilde{f}_k(x_3, x_4, x_6) \text{ are formal series.}$$

Then, $\widetilde{f}_0$ is a formal series first integral of system (9) restricted to $x_5 = 0$. Therefore, from lemma 9, $\widetilde{f}_0 = \widetilde{f}_0(\widetilde{F})$, and hence $f = \widetilde{f}_0(\widetilde{F}) + x_5 h$, with $h = h(x_3, x_4, x_5, x_6)$ a formal series. Furthermore, since $\overline{F} = \widetilde{F} - 2x_5^2$, applying lemma 3 with $x = \overline{F}$ and $y = -2x_5$, we get

$$f = \widetilde{f}_0(\overline{F}) + x_5 h_1, \qquad \text{where} \quad h_1 = h_1(x_3, x_4, x_5, x_6) \text{ a formal series.} \tag{17}$$

Now, we write f as a formal series in the variable x_6, i.e.

$$f = \sum_{k\geqslant 0} \widehat{f}_k x_6^k, \qquad \text{where} \quad \widehat{f}_k = \widehat{f}_k(x_3, x_4, x_5) \text{ are formal series.}$$

By lemma 10, $\widehat{f}_0 = \widehat{f}_0(\widehat{F})$. Moreover, since $\overline{F} = \widehat{F} + s x_6^2$, by lemma 3 with $x = \widehat{F}$ and $y = s x_6^2$, we get

$$f = \widehat{f}_0(\overline{F}) + x_6 h_2, \qquad \text{where} \quad h_2 = h_2(x_3, x_4, x_5) \text{ a formal series.} \tag{18}$$

Imposing that the two equations for f, (17) and (18), must be equal and restricting them to $x_5 = x_6 = 0$, we obtain $\widehat{f}_0(x_3(2x_4 - x_3)) = \widetilde{f}_0(x_3(2x_4 - x_3))$; i.e., $\widehat{f}_0 = \widetilde{f}_0$. Thus,

$x_5h_1 = x_6h_2$. This implies that there exists a formal series $h_3 = h_3(x_3, x_4, x_5, x_6)$ such that $h_1 = x_6h_3$ and $h_2 = x_5h_6$. Then, from (17)

$$f = \widetilde{f}_0(\overline{F}) + x_5x_6h_3. \tag{19}$$

The proof of the proposition will follow from (19) if we show that $h_3 = 0$. We will proceed by contradiction. We assume $h_3 \neq 0$ and we consider two different cases.

Case 1. h_3 is not divisible by x_5x_6. In this case since $\overline{F}$ and f are formal series first integrals of system (9), we get that h_3 is also a formal series first integral. Therefore, h_3 satisfies, after dividing by x_5x_6, (13) with $l = 1$. Then, from proposition 12 we get that h_3 is divisible by x_5x_6, a contradiction.

Case 2. h_3 is divisible by x_5x_6. In this case, we write $h_3 = (x_5x_6)^m h_4$ with $m \geqslant 1$ and $h_4 \neq 0$ is a formal series which is not divisible by x_5x_6. Then, imposing that f is a formal series first integral of system (9), we get that h_4 satisfies (13) with $l = m$. Then, from proposition 12 we get that h_4 is divisible by x_5x_6, a contradiction. □

4. Proof of the main results

Now, we will prove our main results. We first state and prove some preliminary results.

Proposition 13. *Let f be a formal series first integral of system (4) invariant by τ and σ. Then,*

$$f = \sum_{k,l\geqslant 0} c_{k,l} G^k F^l + x_1 h, \tag{20}$$

where $c_{k,l}$ are constants, F and G are introduced in (5) and (6), and h is a formal series in the variables $x_1, \ldots, x_6$.

Proof. First, we claim that any formal series first integral of system (4) restricted to $x_1 = 0$ can be written as

$$f = \sum_{k,l\geqslant 0} c_{k,l} x_2^{2k} \overline{F}^l, \qquad \text{where} \quad c_{k,l} \text{ are constants.} \tag{21}$$

To prove the claim (21), we write any formal series first integral of system (4) restricted to $x_1 = 0$, f, in formal series in the variable x_2 as

$$f = \sum_{k\geqslant 0} f_k x_2^k, \qquad f_k = f_k(x_3, x_4, x_5, x_6) \text{ are formal series.} \tag{22}$$

Each coefficient f_k is a first integral of system (4) restricted to $x_1 = x_2 = 0$ (that is of system (9). Indeed, using that f and x_2 are first integrals of system (4) restricted to $x_1 = 0$, we get that f_0 is a first integral. Now, we proceed by induction, we assume $f_l, 0 \leqslant l \leqslant j$ is a first integral of system (9) and we will prove it for $l = j + 1$. By induction hypothesis and since f and x_2 are first integrals we get

$$0 = \sum_{l\geqslant j+1} \frac{\mathrm{d}f_l}{\mathrm{d}t} x_2^l = x_2^{j+1} \sum_{l\geqslant j+1} \frac{\mathrm{d}f_l}{\mathrm{d}t} x_2^{l-j-1}, \tag{23}$$

where the derivative is evaluated along a solution of system (9). Restricting (23) to $x_2 = 0$, after simplifying by x_2^{j+1}, we get $\mathrm{d}f_{j+1}/\mathrm{d}t = 0$, that is, f_{j+1} is a first integral of system (9). Thus, by induction we have proved that each coefficient f_k is a first integral of system (9).

From proposition 8, we have that $f_k = f_k(\overline{F})$ for $k \geqslant 0$ is a formal series in the variable $\overline{F}$. Thus, from (22), we get

$$f = \sum_{k,l \geqslant 0} f_{k,l} x_2^k \overline{F}^l. \tag{24}$$

Furthermore, since f is invariant by σ, from proposition 5, f only contain monomials $x_1^{l_1} x_2^{l_2} x_3^{l_3} x_4^{l_4} x_5^{l_5} x_6^{l_6}$ with $l_2 + l_3 + l_4$ even. Since $\overline{F}$ only contain monomials $x_3^{l_3} x_4^{l_4} x_5^{l_5} x_6^{l_6}$ with $l_3 + l_4$ even, it follows from (24) that k must be even. This finishes the proof of the claim (21).

Let now f be a formal series first integral of system (4). Then, we write f in formal series in the variable x_1 as

$$f = \sum_{k \geqslant 0} f_k x_1^k, \qquad f_k = f_k(x_2, x_3, x_4, x_5, x_6) \text{ are formal series.}$$

Clearly, f_0 is a formal series first integral of system (4) restricted to $x_1 = 0$. From (21) we get that $f_0 = \sum_{k,l \geqslant 0} c_{k,l} x_2^{2k} \overline{F}^l$. We substitute $\overline{F} = F - sax_1^2$ and $x_2^2 = G + x_1 x_6$ into f_0 and we get

$$f_0 = \sum_{k,l \geqslant 0} c_{k,l} G^k F^l + x_1 g, \qquad \text{with} \quad g = g(x_1, x_2, x_3, x_4, x_5, x_6)$$

a formal series. Then, we can write f as

$$f = \sum_{k,l \geqslant 0} c_{k,l} G^k F^l + x_1 h, \qquad \text{with} \quad h = h(x_1, x_2, x_3, x_4, x_5, x_6)$$

a formal series equal to $g + \sum_{k \geqslant 1} f_k x_1^{k-1}$. Thus, the proposition is proved. □

We write the formal series h appearing in proposition 13 in series in the variable x_1 as

$$h = \sum_{k \geqslant 0} h_k x_1^k, \qquad h_k = h_k(x_2, x_3, x_4, x_5, x_6) \text{ are formal series.}$$

Since f, F and G are first integrals for system (4), taking derivatives with respect to t in (20) we have that, after dividing by x_1, h must satisfy

$$x_1 x_3 \frac{\partial h}{\partial x_1} + x_1 x_5 \frac{\partial h}{\partial x_2} + \left[(x_4 - x_3)x_3 - 2x_5^2\right] \frac{\partial h}{\partial x_3} + \left[-(x_4 - x_3)^2 + s\left(-ax_1^2 + x_6^2\right)\right] \frac{\partial h}{\partial x_4}$$
$$+ [sx_2 x_6 + (x_3 - x_4)x_5] \frac{\partial h}{\partial x_5} + [2x_2 x_5 - x_3 x_6] \frac{\partial h}{\partial x_6} = -x_3 h. \tag{25}$$

Then, evaluating (25) on $x_1 = 0$, we obtain that h_0 satisfies

$$\left[(x_4 - x_3)x_3 - 2x_5^2\right] \frac{\partial h_0}{\partial x_3} + \left[-(x_4 - x_3)^2 + sx_6^2\right] \frac{\partial h_0}{\partial x_4}$$
$$+ [sx_2 x_6 + (x_3 - x_4)x_5] \frac{\partial h_0}{\partial x_5} + [2x_2 x_5 - x_3 x_6] \frac{\partial h_0}{\partial x_6} = -x_3 h_0. \tag{26}$$

Lemma 14. *The unique formal series $h_0 = h_0(x_2, x_3, x_4, x_5, x_6)$ invariant by τ and σ satisfying (26) is $h_0 = 0$.*

For clarity, in the proof of lemma 14, we will state and prove an auxiliary result that will be used therein.

Lemma 15. *Let $g = g(x_3, x_4, x_5, x_6)$ be a formal series invariant by τ and σ satisfying*

$$\left[(x_4 - x_3)x_3 - 2x_5^2\right] \frac{\partial g}{\partial x_3} + \left[-(x_4 - x_3)^2 + sx_6^2\right] \frac{\partial g}{\partial x_4} + (x_3 - x_4)x_5 \frac{\partial g}{\partial x_5} - x_3 x_6 \frac{\partial g}{\partial x_6} = -x_3 g. \tag{27}$$

Then, $g = 0$.

Proof. We will first prove that g is divisible by x_6. We write g in formal series in the variable x_6 as

$$g = \sum_{k\geqslant 0} g_k x_6^k, \qquad g_k = g_k(x_3, x_4, x_5) \text{ are formal series.}$$

We will show that $g_0 = 0$. Before proving it, we show that this will finish the proof of the lemma. Indeed, if $g_0 = 0$, then $g = x_6 h$ where $h = h(x_3, x_4, x_5, x_6)$ is a formal series. Then, imposing that g satisfies (27) we obtain that h satisfies

$$\left[(x_4 - x_3)x_3 - 2x_5^2\right]\frac{\partial h}{\partial x_3} + \left[-(x_4 - x_3)^2 + s x_6^2\right]\frac{\partial h}{\partial x_4} + (x_3 - x_4)x_5\frac{\partial h}{\partial x_5} - x_3 x_6\frac{\partial h}{\partial x_6} = 0.$$

Thus, h is a formal series first integral of system (9). From proposition 8 we get that $h = h(\overline{F})$ is a formal series in $\overline{F}$ and then, $g = x_6 h(\overline{F})$. Furthermore, since g is invariant by τ_1, from proposition 5 (restricted to $x_1 = x_2 = 0$) we get that g must contain monomials of the form $x_3^{l_3}x_4^{l_4}x_5^{l_5}x_6^{l_6}$ with $l_5 + l_6$ even. This is impossible unless $g = 0$, because $\overline{F}$ only contain monomials of the form $x_3^{l_3}x_4^{l_4}x_5^{l_5}x_6^{l_6}$ with $l_5 + l_6$ even.

In short, to prove the lemma it remains to prove that $g_0 = 0$. We write g_0 in formal series in the variable x_5 as

$$g_0 = \sum_{l\geqslant 0} g_{0,l} x_5^l, \qquad g_{0,l} = g_{0,l}(x_3, x_4) \text{ are formal series.} \tag{28}$$

From proposition 5 on $x_1 = x_2 = x_6 = 0$ we get that g_0 only contains monomials $x_3^{l_3}x_4^{l_4}x_5^{l_5}$ with l_5 even.

Restricting (27) to $x_5 = x_6 = 0$ we get that $g_{0,0}$ must satisfy

$$(x_4 - x_3)\left[x_3\frac{\partial g_{0,0}}{\partial x_3} - (x_4 - x_3)\frac{\partial g_{0,0}}{\partial x_4}\right] = -x_3 g_{0,0}. \tag{29}$$

So, $g_{0,0} = (x_4 - x_3)f_1(x_3, x_4)$. Substituting $g_{0,0}$ into (29) and simplifying we obtain that f_1 must satisfy

$$x_3\frac{\partial f_1}{\partial x_3} - (x_4 - x_3)\frac{\partial f_1}{\partial x_4} = f_1.$$

The general solution of this linear partial differential equation is $f_1 = x_3 f_2(x_3(2x_4 - x_3))$ where f_2 is an arbitrary function, for us a formal series in the variable $x_3(2x_4 - x_3)$. So, we can write

$$g_{0,0} = x_3(x_3 - x_4)\sum_{k\geqslant 0} c_k (x_3(2x_4 - x_3))^k.$$

Since $x_3(2x_4 - x_3) = \widehat{F} + 2x_5^2$, we get that $g_{0,0} = x_3(x_3 - x_4)\sum_{k\geqslant 0} c_k \widehat{F}^k + x_5^2 f_3(x_3, x_4, x_5)$. Now, using the formal series (28) and the fact that its power series in x_5 are even, we obtain

$$g_0 = x_3(x_3 - x_4)\sum_{k\geqslant 0} c_k \widehat{F}^k + x_5^2 f_4, \qquad f_4 = f_4(x_3, x_4, x_5) \qquad \text{are formal series.} \tag{30}$$

Now we consider $g_0 \neq 0$ and we will reach a contradiction. We consider two cases.

Case 1. g_0 is not divisible by $\widehat{F}$. Restricting g_0 to the invariant set $\{\hat{F} = 0\}$ or equivalently $\left\{2x_5^2 = x_3(2x_4 - x_3)\right\}$, and then imposing that g_0 satisfies (27) restricted to $x_6 = 0$ we get, after dividing by x_5^2 and taking into account that $\hat{F}$ is a formal series first integral of this system, that

$$-2(2x_3 - x_4)c_0 - x_3 x_4\frac{\partial \overline{f}_4}{\partial x_3} - (x_4 - x_3)^2\frac{\partial \overline{f}_4}{\partial x_4} = (2x_4 - 3x_3)\overline{f}_4, \tag{31}$$

where $\overline{f}_4 = \overline{f}_4(x_3, x_4)$ denotes the restriction of f_4 to $\{\hat{F} = 0\}$. Now, if we restrict (31) to $x_3 = 0$ and denote by $\widetilde{f}_4$ the restriction of $\overline{f}_4$ to $x_3 = 0$, then

$$2c_0 - x_4 \frac{\mathrm{d}\widetilde{f}_4}{\mathrm{d}x_4} = 2\widetilde{f}_4.$$

The general solution of this differential equation is $\widetilde{f}_4(x_4) = c_0 + c_1/x_4^2$. Since $\widetilde{f}_4$ is a formal series, we get $\widetilde{f}_4 = c_0$. Then, from lemma 4, $\overline{f}_4 = c_0 + x_3 f_5$, for some formal series $f_5 = f_5(x_3, x_4)$. In an analogous way, restricting equation (31) to $x_3 = 2x_4$, and denoting $\widehat{f}_4$ the restriction of $\overline{f}_4$ to $x_3 = 2x_4$, we obtain

$$-6c_0 - x_4 \frac{\mathrm{d}\widehat{f}_4}{\mathrm{d}x_4} = -4\widehat{f}_4, \qquad \text{which implies} \quad \widehat{f}_4 = \frac{3}{2}c_0 + c_1 x_4^4 \text{ with } c_1 \text{a constant.}$$

Then, from lemma 4, we get $\overline{f}_4 = 3c_0/2 + c_1 x_4^4 + (x_3 - 2x_4) f_6$ for some formal series $f_6 = f_6(x_3, x_4)$. The two equations for $\overline{f}_4$ on $x_3 = x_4 = 0$ imply that $c_0 = 0$ and thus $\overline{f}_4 = x_3 f_5$. Now, we will prove that $\overline{f}_4 = 0$. To do it, we assume $\overline{f}_4 \neq 0$ and we will reach a contradiction. We write $\overline{f}_4 = x_3^m h$ where $m \geqslant 1$ and h is a formal series which is not divisible by x_3 and $\overline{f}_4$ satisfies (31) with $c_0 = 0$, i.e., after dividing by x_3^m

$$-x_3 x_4 \frac{\partial h}{\partial x_3} - (x_4 - x_3)^2 \frac{\partial h}{\partial x_4} = ((2 + m)x_4 - 3x_3)h. \tag{32}$$

Then, if we write $h = \sum_{l \geqslant 0} h_l x_3^l$, with $h_l = h_l(x_4)$ a formal series, then $h_0 \neq 0$ satisfies (32) evaluated on $x_3 = 0$, i.e.,

$$-x_4^2 \frac{\mathrm{d}h_0}{\mathrm{d}x_4} = (2 + m)x_4 h_0.$$

Its general solution is $h_0 = c_2/x_4^{m+2}$. Taking into account that h_0 is a formal series, implies $h_0 = 0$, a contradiction.

In short, $\overline{f}_4 = 0$ which by lemma 4 implies that $f_4 = \widehat{F} f_7$ for some formal series $f_7 = f_7(x_3, x_4, x_5)$. Then, from (30) and since $c_0 = 0$, we get

$$g_0 = \widehat{F}\left(x_3(x_3 - x_4) \sum_{k \geqslant 1} c_k \widehat{F}^{k-1} + x_5^2 f_7 \right),$$

a contradiction with the fact that g_0 is not divisible by $\widehat{F}$.

Case 2. g_0 is divisible by $\widehat{F}$. In this case we write $g_0 = \widehat{F}^m h$ for some $m \geqslant 1$ and $h = h(x_3, x_4)$ a formal series not divisible by $\widehat{F}$. Since $\widehat{F}$ is a first integral of system (9) restricted to $x_6 = 0$, we have that h satisfies the same equation as g_0. Then, proceeding as in case 1, we reach a contradiction. □

Proof of lemma 14. We decompose h_0 in formal series in the variable x_2 as

$$h_0 = \sum_{k \geqslant 0} g_k x_2^k, \qquad g_k = g_k(x_3, x_4, x_5, x_6) \text{ are formal series.} \tag{33}$$

Then, we will prove by induction that

$$g_k = 0, \qquad \text{for} \quad k \geqslant 0. \tag{34}$$

Clearly g_0 satisfies (26) restricted to $x_2 = 0$, that is, (27). By lemma 15, we get that $g_0 = 0$ and (34) is proved for $k = 0$. Now, we assume that the claim (34) is true for $k = 0, \ldots, m - 1$ ($m \geqslant 1$) and we will prove it for $k = m$. By the induction hypothesis, we get that $g_m + g_{m+1}x_2 + g_{m+2}x_2^2 + \cdots$ satisfies (26) replacing h_0 (after dividing by x_2^m). Taking $x_2 = 0$ we obtain that g_m satisfies (27). Then, from lemma 15, we get that $g_m = 0$, and prove the claim (34) for $k = m$. Hence, the claim (34) is proved. Therefore, using (33) we get that $h_0 = 0$ and finish the proof of the lemma. □

Proof of theorem 1. Let g be a formal series first integral of system (4). If g is a formal series first integral in the variables F and G the theorem is proved. So, we can assume that g is not a formal series in the variables F and G. Moreover, without loss of generality the formal series g has no independent term. We also can assume that g is not divisible by any formal series T depending only on F and G; otherwise if $T(F, G)$ divides g, then we can take $g/T(F, G)$ instead of g a new first integral.

By proposition 5, we have that

$$f = (g \cdot \tau(g)) \cdot \sigma(g \cdot \tau(g)) \tag{35}$$

is also a first integral of system (4) invariant by τ and σ. We first prove that f is a formal series in the variables F and G. From proposition 13 and lemma 14, we have that f can be written as

$$f = \sum_{k,l \geqslant 0} c_{k,l} G^k F^l + x_1 h, \qquad h = \sum_{k \geqslant 1} h_k x_1^k, \qquad h_k = h_k(x_2, x_3, x_4, x_5, x_6),$$

where h_k are formal series in their variables. Since f, G and F are formal series first integrals of system (4), we obtain that the coefficient of x_1^2 in (7) provides the equality

$$\begin{aligned}
&\left[(x_4 - x_3)x_3 - 2x_5^2\right]\frac{\partial h_1}{\partial x_3} + \left[-(x_4 - x_3)^2 + s x_6^2\right]\frac{\partial h_1}{\partial x_4} \\
&\qquad + [s x_2 x_6 + (x_3 - x_4)x_5]\frac{\partial h_1}{\partial x_5} + [2x_2x_5 - x_3x_6]\frac{\partial h_1}{\partial x_6} = -2x_3 h_1.
\end{aligned}$$

In a similar way, the coefficient of x_1^3 in (7) provides the equality

$$\begin{aligned}
&x_5\frac{\partial h_1}{\partial x_2} + \left[(x_4 - x_3)x_3 - 2x_5^2\right]\frac{\partial h_2}{\partial x_3} + \left[-(x_4 - x_3)^2 + s x_6^2\right]\frac{\partial h_2}{\partial x_4} \\
&\qquad + [s x_2 x_6 + (x_3 - x_4)x_5]\frac{\partial h_2}{\partial x_5} + [2x_2x_5 - x_3x_6]\frac{\partial h_2}{\partial x_6} = -3x_3 h_2.
\end{aligned}$$

Finally, the coefficient of x_1^{k+1} in (7) with $k \geqslant 3$ provides the equality

$$\begin{aligned}
&x_5\frac{\partial h_{k-1}}{\partial x_2} + \left[(x_4 - x_3)x_3 - 2x_5^2\right]\frac{\partial h_k}{\partial x_3} + \left[-(x_4 - x_3)^2 + s x_6^2\right]\frac{\partial h_k}{\partial x_4} - as\frac{\partial h_{k-2}}{\partial x_4} \\
&\qquad + [s x_2 x_6 + (x_3 - x_4)x_5]\frac{\partial h_k}{\partial x_5} + [2x_2x_5 - x_3x_6]\frac{\partial h_k}{\partial x_6} = -(k+1)x_3 h_k.
\end{aligned} \tag{36}$$

We claim that

$$h_k = 0 \qquad \text{for} \quad k \geqslant 1. \tag{37}$$

Clearly h_1 satisfies equation (26) with h_0 replaced by h_1 and the right-hand side replaced by $-2x_3h_1$. The arguments used for proving that $h_0 = 0$ in lemma 14 imply that $h_1 = 0$ and finish the proof of (37) for $k = 1$. Now, assume (37) is proved for $k = 1, \ldots, m-1$ ($m \geqslant 2$) and we want to prove it for $k = m$. By the induction hypothesis equation (36) with $k = m$ becomes (26) with h_0 replaced by h_m and the right-hand side replaced by $-(m+1)x_3h_m$. Then, the same arguments used for proving that $h_0 = 0$ in lemma 14 imply that $h_m = 0$ and proves (37) for $k = m$. Hence, by induction arguments, the claim (37) is proved for $k \geqslant 1$. Therefore, $h = 0$ and thus,

$$f = \sum_{k+l>0} c_{k,l} G^k F^l. \tag{38}$$

Hence, from (35) we get that f must be reducible, that is, there exist formal series $T = T(F, G)$ and $T_1 = T_1(F, G)$ such that $f = T(F, G)T_1(F, G)$. Furthermore, we can assume that T is irreducible. Then, from (35) we get that $T(F, G)$ divides $g \cdot \tau(g)$ or $\sigma(g \cdot \tau(g))$. In the first case, we also can assume that divides $\tau(g)$; otherwise we reach a contradiction with the assumptions on g. However, if $T(F, G)$ divides $\tau(g)$, then $\tau(g) = T(F, G)T_2$ for some formal series $T_2 = T_2(x_1, x_2, x_3, x_4, x_5, x_6)$, and thus

$$g = \tau^2(g) = \tau(T(F, G))\tau(T_2),$$

a contradiction with the assumptions on g.

Now, we assume that $T(F, G)$ divides $\sigma(g \cdot \tau(\sigma))$. With similar arguments to those used for the case in which $T(F, G)$ divides $g \cdot \tau(\sigma)$, we reach a contradiction with the assumptions on g. So, the theorem is proved. □

Proof of theorem 2. Since F and G are analytic first integrals of system (4), it is clear that any analytic function in a neighbourhood of zero in the variables F and G is an analytic first integral of system (4) in a neighbourhood of zero. To prove that these are the only ones, we proceed by contradiction. Assume that g is an analytic first integral of system (4) which is not a function of F, G. Then, there exists a neighbourhood $U \subset \mathbb{R}^6$ of the origin such that $g_{|U}$ is a nontrivial first integral of system (4). Clearly, $g_{|U}$ can be written as a formal series which turns out to be convergent. Hence, in U, g is a formal series first integral which cannot be written as a formal series in the variables F and G, a contradiction with theorem 1. Thus, theorem 2 is proved. □

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